CONSTRUCTION PROCESS FOR SIMPLE LIE ALGEBRAS

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Abstract

We give here a construction process for the complex simple Lie algebras and the non Hermitian type real forms which intersect the minimal nilpotent complex adjoint orbit, using a finite dimensional irreducible representation of the conformal group, or of some 2-fold covering of it, with highest weight a semi-invariant of degree 4. This process leads to a 5-graded simple complex Lie algebra and the underlying semi-invariant is intimately related to the structure of the minimal nilpotent orbit. We also describe a similar construction process for the simple real Lie algebras of Hermitian type.

Conformal and Meta-Conformal Groups

Let V be a Euclidean complex vector space and Q a degree 4 homogeneous polynomial on V. Let L be defined by

$$L := \{g \in GL(V) \mid \exists \gamma(g) \in \mathbb{C}, Q(gz) = \gamma(g)Q(z)\}$$

and suppose that it has an open orbit and that L is self-ajoint :

$$\forall g \in L, g^* \in L.$$

More precisely, V is a semi-simple complex Jordan algebra with rank ≤ 4, L is the structure group of V and Q semi-invariant for L.

If $V = \sum_{i=1}^{s} V_i$ is the decomposition of V into simple ideals, then $Q(z) = \prod_{i=1}^{s} \Delta_i^{k_i}(z_i)$, where Δ_i is the determinant polynomilal of V_i and the k_i are positive integers such that $\sum_{i=1}^{s} k_i r_i = 4$, r_i being the rank of the Jordan algebra V_i. In the sequel, e is the unit element of V.

Let K be the conformal group of V, that is the set of rationnal transformations g of V such that, for each $z \in V$ where g is defined, the differential $Dg(z) \in L$.

K is a semi-simple Lie group. It is generated by L, the group N of translations $\tau_a (a \in V)$ and the conformal inversion $\sigma(z) = \nabla log Q(z)$.

$P := L \ltimes N$ is a maximal parabolic subgroup of K and L is its Lévi factor.

The Lie algebra $\mathfrak{k}$ of K writes $\mathfrak{k} = \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1$, with

$$\mathfrak{k}_{-1} = Lie(N), \mathfrak{k}_1 = Lie(\sigma N \sigma)\ , \mathfrak{k}_0 = Lie(L).$$

Let $\mathfrak{p}$ be the complex vector space generated by the polynomials $Q(z-a)$ with $a \in V$.

We first suppose that there exists a character χ of L such that

$$Q(l.z) = \chi(l)^2 Q(z).$$

Then, the conformal group K acts on $\mathfrak{p}$ by :

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$$(\kappa(g)p)(z) = \mu(g,z)p(g^{-1}z)$$

with

$$\mu(g^{-1},z) = \chi(Dg(z))^{-1}.$$

In particular

$$(\kappa(\tau_a)p)(z) = p(z-a) \qquad (a \in V)$$
$$(\kappa(l)p)(z) = \chi(l)p(l^{-1}z) \qquad (l \in L)$$
$$(\kappa(\sigma)p)(z) = Q(z)p(-z^{-1}).$$

κ is the finite dimensional irreducible representation of K with highest weight Q. It is also the representation $Ind_P^K\chi$ obtained by parabolic induction from the chatacter χ of L. The derived representation $d\kappa$ can be obtained on the generators of $\mathfrak{k}$ as follows : For $X \in \mathfrak{k}_{-1}$, let $v \in V$ be such that $exp(tX) = \tau_{tv}$, then

$$(d\kappa(X)p)(z) = \tfrac{d}{dt}\mid_{t=0} p(z-tv) = -Dp(z)(v).$$

For $X \in \mathfrak{k}_0$,

$$(d\kappa(X)p)(z) = \frac{d}{dt}\mid_{t=0} \gamma(exp(\frac{t}{2}X))p(exp(-tX)z)$$
$$= D\gamma(id)\circ D(exp)(0)(\frac{1}{2}X)p(z) - Dp(z)(D(exp)(0)(X)z)$$
$$= \frac{1}{2}D\gamma(id)(X)p(z) - Dp(z)(X.z).$$

For $X \in \mathfrak{k}_1$, $exp(tX) = \sigma\tau_{tu}\sigma$ with $u \in V$, then

$$(d\kappa(X)p)(z) = \frac{d}{dt}\mid_{t=0} (\kappa(\sigma\tau_{tu}\sigma)p)(z)$$
$$= \frac{d}{dt}\mid_{t=0} (\kappa(\sigma)\kappa(\tau_{tu})[Q(z)p(\sigma(z))] = \frac{d}{dt}\mid_{t=0} (\kappa(\sigma)[Q(z-tu)p(\sigma(z-tu))]$$
$$= \frac{d}{dt}\mid_{t=0} [Q(z)Q(\sigma(z)-tu)p(\sigma(\sigma(z)-tu))] = \frac{d}{dt}\mid_{t=0} Q(e-tzu)p(z\sigma(e-tzu))$$
$$= -DQ(e)(zu)p(z) - Dp(z)(zu)$$

If the character χ of L does'nt exist, we consider the 2-fold-covering group of L defined by

$$\tilde{L}^{(2)} = \{(l,\alpha) \in L \times \mathbb{C}^* \mid \alpha^2 = \gamma(l)^{-1}\}.$$

$\tilde{L}^{(2)}$ acts on V by $(l,\alpha).z = \alpha l.z$, then

$$Q((l,\alpha).z) = \alpha^4\gamma(l)Q(z) = \alpha^2 Q(z) = \tilde{\chi}(l,\alpha)^2 Q(z)$$

where $\tilde{\chi}(l,\alpha) = \alpha$ is a character of $\tilde{L}^{(2)}$. Moreover, $\tilde{L}^{(2)}$ is a subgroup of a 2-fold-covering group of the conformal group K, which will be called the meta-conformal group, defined by :

$$\tilde{K}^{(2)} = \{(g,\phi) \in K \times \mathcal{O}(V) \mid \phi(z)^2 = \gamma(Dg(z))^{-1}\}$$

equipped with the group law given by :

$$(g_1,\phi_1).(g_2,\phi_2) = (g_1g_2,\phi_3), \text{ with } \phi_3(z) = \phi_1(g_2z)\phi_2(z).$$

Proposition. For each $g \in K$, the function $\psi_g : z \mapsto \gamma(Dg(z))^{-1}$, holomorphic on $V - \{z \in V \mid Dg(z) = 0\}$, has an analytic continuation to V. Moreover, there exists $\phi_g \in O(V)$, such that $\psi_g = \phi_g^2$.

Proof. Using the cocycle property $D(g_1g_2)(z) = Dg_1(g_2(z))Dg_2(z)$ which implies $\psi_{g_1g_2}(z) = \psi_{g_1}(g_2.z)\psi_{g_2}(z)$, it suffises to prove the proposition for the generators of K. For $g = l \in L$, $Dg(z) = l$ and $\psi_g(z) = \gamma(l)^{-1}$ for $z \in V$. For $g = \tau_v$, $Dg(z) = id_V$ and $\psi_g(z) = 1$ for $z \in V$. If $g = \sigma$, as for each invertible z,$D\sigma(z) = P(z)^{-1}$ and $det(P(z)w) = det(z)^2det(w)$ (cf. [F-K] Proposition II.3.3), then we get $\psi_g(z) = Q(z)^2$ which is a polynomial, it follows that ψ_g has analytic continuation to V and $\phi_g(z)^2 = \psi$, ϕ_g being the holomorphic function $\phi_g(z) = Q(z)$. $\square$

Corollary. $\tilde{K}^{(2)}$ is a 2-fold covering group of K, which contains the covering $\tilde{L}^{(2)}$ of L. Moreover, $\tilde{K}^{(2)}$ is generated by the elements

$$(l, \alpha) \quad \text{with } (l \in L, \alpha = \gamma(l)^{-\frac{1}{2}}),$$
$$(\tau_v, 1) \quad \text{with } v \in V,$$
$$(\sigma, Q).$$

The group $\tilde{K}^{(2)}$ will be called the meta-conformal group of V, associated to the semi-invariant Q. The subgroup $\tilde{P}^{(2)}$ generated by $\tilde{L}^{(2)}$ and the $(\tau_v, 1)$ is maximal parabolic of $\tilde{K}^{(2)}$ with Levi factor $\tilde{L}^{(2)}$.

We consider the representation $\tilde{\kappa}$ of $\tilde{K}^{(2)}$ in $\mathfrak{p}$ defined by

$$(\tilde{\kappa}(\tau_v, 1)p)(z) = p(z - v)$$
$$(\tilde{\kappa}(l, \alpha)p)(z) = \alpha^{-1}p(l^{-1}.z)$$
$$(\tilde{\kappa}(\sigma, Q)p)(z) = Q(z)p(-\sigma(z)).$$

$\tilde{\kappa}$ is the finite dimensional irreducible representation of $\tilde{K}^{(2)}$ with highest weight Q. It is also the representation $Ind_{\tilde{P}^{(2)}}^{\tilde{K}^{(2)}}\tilde{\chi}$ obtained by parabolic induction from the chatacter $\tilde{\chi}$ of $\tilde{L}^{(2)}$. The derived representation $d\tilde{\kappa}$ can be obtained on the generators of $\mathfrak{k}$ as follows :

For $X \in \mathfrak{k}_{-1}$, let $v \in V$ be such that $exp(tX) = (\tau_{tv}, 1)$, then

$$(d\tilde{\kappa}(X)p)(z) = \tfrac{d}{dt} |_{t=0}\, p(z - tv) = -Dp(z)(v).$$

For $X \in \mathfrak{k}_0$,

$$(d\tilde{\kappa}(X)p)(z) = \frac{d}{dt} |_{t=0}\, \gamma(exp(\frac{t}{2}X))p(exp(-tX)z)$$

$$= D\gamma(id) \circ D(exp)(0)(\frac{1}{2}X)p(z) - Dp(z)(D(exp)(0)(X)z)$$

$$= \frac{1}{2}D\gamma(id)(X)p(z) - Dp(z)(X.z).$$

For $X \in \mathfrak{k}_1$,

$$\begin{aligned}
(d\tilde{\kappa}(X)p)(z) &= \frac{d}{dt} \mid_{t=0} (\tilde{\kappa}((\sigma,Q).(\tau_{tu},1).(\sigma,Q))p)(z) \\
&= \frac{d}{dt} \mid_{t=0} (\tilde{\kappa}((\sigma,Q))\tilde{\kappa}((\tau_{tu},1))\tilde{\kappa}((\sigma,Q))p)(z) \\
&= \frac{d}{dt} \mid_{t=0} (\tilde{\kappa}((\sigma,Q))\tilde{\kappa}((\tau_{tu},1))[Q(z)p(\sigma(z))] \\
&= \frac{d}{dt} \mid_{t=0} (\tilde{\kappa}((\sigma,Q))[Q(z-tu)p(\sigma(z-tu))] \\
&= \frac{d}{dt} \mid_{t=0} [Q(z)Q(\sigma(z)-tu)p(\sigma(\sigma(z)-tu))] \\
&= \frac{d}{dt} \mid_{t=0} Q(e-tzu)p(z\sigma(e-tzu)) \\
&= -DQ(e)(zu)p(z) - Dp(z)(zu).
\end{aligned}$$

Notice that the infinitesimal representations $d\kappa$ and $d\tilde{\kappa}$ are equal. In the sequel, We denote this representation of $\mathfrak{k}$ in $\mathfrak{p}$ by ρ.

Graduation of $\mathfrak{k}$ and $\mathfrak{p}$.

The Lie algebra $\mathfrak{k} = Lie(K)$ is the Kantor-Koecher-Tits Lie algebra of V.

We denote by h_t the dilation of V : $h_t.z = e^{-t}z \quad (t \in \mathbb{R})$. Then $h_t \in L, h_t = e^{tH}$ with $H \in Lie(L)$ and $\chi(h_t) = e^{-2t}$ (In the case of the character $\tilde{\chi}$ of $\tilde{L}^{(2)}$, we consider $\tilde{h}_t = (h_t, e^{2t}) \in \tilde{L}^{(2)}$ in such a way that $\tilde{\chi}(\tilde{h}_t) = e^{-2t}$).

We can prove that $\rho(H) = \mathcal{E} - 2$, where $\mathcal{E}$ is the Euler operator $(\mathcal{E}p)(z) =< z, \nabla p(z) >$.

H defines a graduation of $\mathfrak{k}$:

$$\mathfrak{k} = \mathfrak{k}_{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1$$

with

$$\mathfrak{k}_j = \{X \in \mathfrak{k} \mid ad(H)X = jX\} \qquad j = -1, 0, 1.$$

Notice that

$$Ad(\sigma) : \mathfrak{k}_j \to \mathfrak{k}_{-j}, X \mapsto \sigma X \sigma$$
$$\mathfrak{k}_{-1} = Lie(N) \simeq V, \mathfrak{k}_0 = Lie(L), \mathfrak{k}_1 = Lie(\sigma N \sigma) \simeq V$$

and that

$$H \in \mathfrak{z}(\mathfrak{k}_0) \quad \text{(centre of } \mathfrak{z}(\mathfrak{k}_0)).$$

H defines also a graduation of $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{p}_{-2} + \mathfrak{p}_{-1} + \mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2$$

with

$$\mathfrak{p}_j = \{p \in \mathfrak{p} \mid \rho(H)p = jp\}.$$

$\mathfrak{p}_j$ is the set of homogeneous polynomials of degree $j+2$ in $\mathfrak{p}$.

Notice that

$$\kappa(\sigma) : \mathfrak{p}_j \to \mathfrak{p}_{-j}, p \mapsto \kappa(\sigma)p$$
$$\mathfrak{p}_{-2} = \mathbb{C}, \mathfrak{p}_2 = \mathbb{C}.Q, \mathfrak{p}_{-1} \simeq V, \mathfrak{p}_1 \simeq V.$$

Construction process of simple Lie algebras

Let $\mathfrak{g}$ be the vector space defined by $\mathfrak{g} := \mathfrak{k} + \mathfrak{p}$. Let's denote by $E = Q, F = 1$ $(E, F \in \mathfrak{p})$.

Theorem 1. There exists on $\mathfrak{g}$ a unique Lie algebra structure such that :

(S_1) $[X, X'] = [X, X']_{\mathfrak{k}}$ $(X, X' \in \mathfrak{k})$

(S_2) $[X, p] = \rho(X)p$ $(X \in \mathfrak{k}, p \in \mathfrak{p})$

(S_3) $[E, F] = H$.

Lemma 1.

(a) $\forall X \in \mathfrak{k}_{-1}, \rho(X)F = 0$ et $\forall Y \in \mathfrak{k}_1, \rho(Y)E = 0$.

(b) $\forall X \in \mathfrak{k}_{-1}, \rho(X)E \in \mathfrak{p}_1$ and $\forall Y \in \mathfrak{k}_1, \rho(Y)F \in \mathfrak{p}_{-1}$.

(c) $\forall X \in \mathfrak{k}_0$, $\rho(X)E = \alpha(X)E$ and $\rho(X)F = -\alpha(X)F$ with $\alpha(X) = -\frac{1}{2}D\gamma(id)(X)$.

Proof.

(a) Let be $X \in \mathfrak{k}_{-1}$, then $(\rho(X)F)(z) = -DF(z)(v) = 0$.

Let be $Y \in \mathfrak{k}_1$, then

$$\begin{aligned}(\rho(Y)E)(z) &= \frac{d}{dt}_{t=0}(\kappa(\sigma)\kappa(\tau_{tv})\kappa(\sigma)E)(z)\\ &= \frac{d}{dt}_{t=0}(\kappa(\sigma)\kappa(\tau_{tv})F)(z) = \frac{d}{dt}_{t=0}(\kappa(\sigma)F)(z) = 0.\end{aligned}$$

(b) Let be $X \in \mathfrak{k}_{-1}$. Then for $\lambda \in \mathbb{C}^*$,

$$\begin{aligned}(\rho(X)E)(\lambda z) &= \frac{d}{dt}_{t=0} E(\lambda z + tv) = \frac{d}{dt}_{t=0} E(\lambda(z + \frac{t}{\lambda}v))\\ &= \frac{d}{dt}_{t=0} \lambda^4 E(z + \frac{t}{\lambda}v) = \lambda^4 \frac{1}{\lambda}\frac{d}{ds}_{s=0} E(z + sv)\\ &= \lambda^3(\rho(X)E)(z).\end{aligned}$$

Let be $Y \in \mathfrak{k}_1$. Then $Y = \sigma X \sigma$ with $X \in \mathfrak{k}_{-1}$. It follows that $(\rho(Y)F) = \kappa(\sigma)\rho(X)E \in \kappa(\sigma)(\mathfrak{p}_1) = \mathfrak{p}_{-1}$.

(c) Let be $X \in \mathfrak{k}_0$ then

$$\begin{aligned}(\rho(X)E)(z) &= \frac{d}{dt}_{t=0}\gamma(exp(\frac{t}{2}X)E(exp(-tX)z)\\ &= \frac{d}{dt}_{t=0}\gamma(exp(-\frac{t}{2}X))E(z) = -\frac{1}{2}D\gamma(id)(X)E(z)\end{aligned}$$

$$\begin{aligned}(\rho(X)F)(z) &= \frac{d}{dt}_{t=0}\gamma(exp(\frac{t}{2}X)F(exp(-tX)z)\\ &= \frac{d}{dt}_{t=0}\gamma(exp(\frac{t}{2}X))F(z) = \frac{1}{2}D\gamma(id)(X)F(z). \quad \square\end{aligned}$$

Lemma 2.
(a) $\forall p \in \mathfrak{p}_1, \exists! X \in \mathfrak{k}_{-1}, p = \rho(X)E$.
(b) $\forall p \in \mathfrak{p}_{-1}, \exists! Y \in \mathfrak{k}_1, p = \rho(Y)F$.

Proof. As $\mathfrak{p}$ is a simple $\mathfrak{k}$-module with highetst weight E, we can write $\mathfrak{p} = \rho(\mathcal{U}(\mathfrak{k}))E$, where $\mathcal{U}(\mathfrak{k})$ is the envelopping algebra of $\mathfrak{k}$. This allows, using lemma 1, to prove that X, Y, X_1, X_2 exist. On an other side, the linear maps

$$\mathfrak{k}_{-1} \to \mathfrak{p}_1, X \mapsto \rho(X)E \text{ and } \mathfrak{k}_1 \to \mathfrak{p}_{-1}, Y \mapsto \rho(Y)F$$

are injective. In fact, let $X \in \mathfrak{k}_{-1}$ be such that $d\kappa(X)E = 0$. Then, for each $z \in V$, $\frac{d}{dt}_{t=0}E(z+tv) = 0$. As $E(z) = \prod_{i=1}^{s} \Delta_i^{k_i}(z_i)$, then, denoting by $z = (z_i)$ and $v = (v_i)$, we get

$$\frac{d}{dt}_{t=0}E(z+tv) = \frac{d}{dt}_{|t=0} \prod_{i=1}^{s} \Delta_i^{k_i}(z_i + tv_i) = 0.$$

Then if all the z_i are invertible,

$$\frac{d}{dt}_{t=0} \prod_{i=1}^{s} \Delta_i^{k_i}(z_i + tv_i) = \sum_{i=1}^{s} k_i \Delta_i^{k_i}(z_i) tr(z_i^{-1}u_i) \prod_{j \neq i} \Delta_j^{k_j}(z_j) = E(z) \sum_{i=1}^{s} k_i tr(z_i^{-1}v_i),$$

then for each $z_i \in V_i$ invertible, $\sum_{i=1}^{s} k_i tr(z_i^{-1}v_i) = 0$, which implies that for each $z_i \in V_i$, $\sum_{i=1}^{s} k_i tr(z_i v_i) = 0$, and finally for each $1 \leq i \leq s$, $\forall z_i \in V_i, tr(z_i v_i) = 0$, and as the bilinear form $tr(xy)$ is non degenerated on each V_i, then for each i, $v_i = 0$, i.e. $v = 0$ and $X = 0$.

Let $Y \in \mathfrak{k}_1$ be such that $\rho(Y)F = 0$, i.e. $\kappa(\sigma)d\kappa(X)E = 0$ where $Y = \sigma X \sigma$ with $X \in \mathfrak{k}_{-1}$. Then $\rho(X)E = 0$ and then $X = 0$ and $Y = 0$. $\square$

Lemma 3. The representation $\rho : \mathfrak{k} \to End(\mathfrak{p})$ is injective.

Proof. Let $X \in \mathfrak{k}$ be such that $\rho(X) = 0$. We write $X = X_{-1} + X_0 + X_1$ with $X_j \in \mathfrak{k}_j$. As $\rho(\mathfrak{k}_j)(\mathfrak{p}_i) \subset \mathfrak{p}_{i+j}$, we obtain that $\rho(X_{-1}) = \rho(X_0) = \rho(X_1) = 0$.

$\rho(X_{-1}) = 0$ implies that for all $p \in \mathfrak{p}$ and $z \in V$, $Dp(z)(v) = 0$ (where $exp(tX_{-1}) = \tau_{tv}$). In particular, if $p(z) = tr(zu)$ with $u \in V$, then $tr(vu) = 0$ for arbitrary u, and , as V semisimple means that the bilinear form $tr(uv$ is non degenerate, then we get $v = 0$ and$X_{-1} = 0$.

$\rho(X_0) = 0$ implies $\rho(X_0)E = 0$, then $D\gamma(id)(X_0) = 0$. Now, as for each linear form p on V, $\rho(X_0)p(z) == \frac{1}{2}D\gamma(id)(X_0)p(z) + \frac{d}{dt}_{t=0}p(exp(-tX_0)z) = 0$, and as $D\gamma(id)(X_0) = 0$, we have $\frac{d}{dt}_{t=0}p(exp(-tX_0)z) = p(-X_0.z) = 0$. If $p(z) = tr(zu)$ with $u \in V$ then $tr(-X_0.zu) = 0$ for each $z, u \in V$, then $X_0 = 0$.

As $X_1 = \sigma Y_{-1} \sigma$ with $Y_{-1} \in \mathfrak{k}_{-1}$, then it is clear that $\rho(X_1) = 0$ implies $\rho(Y_{-1}) = 0$, implies $Y_{-1} = 0$ and $X_1 = 0$.

Proof of theorem 1. We define the Lie bracket $[p, p'] \in \mathfrak{k}$ for two elements p, p' of $\mathfrak{p}$ in such a way that

$$[E, F] = H,\ [\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{k}_{i+j}$$

and

$$[X, [p, p']]_{\mathfrak{k}} = [[X, p], p']_{\mathfrak{k}} + [p, [X, p']]_{\mathfrak{k}} = [\rho(X)p, p']_{\mathfrak{k}} + [p, \rho(X)p']_{\mathfrak{k}}. \quad (\forall X \in \mathfrak{k}).$$

It follows that

$$\forall X \in \mathfrak{k}_{-1}, [\rho(X)E, F] = [X, [E, F]] = [X, H] = X$$

and the bracket $[\mathfrak{p}_1, \mathfrak{p}_{-2}] \subset \mathfrak{k}_{-1}$ is well defined.

$$\forall Y \in \mathfrak{k}_1, [\rho(Y)F, E] = [Y, [F, E]] = -[Y, H] = Y$$

and the bracket $[\mathfrak{p}_{-1}, \mathfrak{p}_2] \subset \mathfrak{k}_1$ is well defined.

$$\forall X \in \mathfrak{k}_{-1}, p \in \mathfrak{p}_0, [\rho(X)E, p] = [[X, E], p] = [X, [E, p]] - [E, [X, p]] = [\rho(X)p, E] \in [\mathfrak{p}_{-1}, \mathfrak{p}_2]$$

and the bracket $[\mathfrak{p}_1, \mathfrak{p}_0] \subset \mathfrak{k}_1$ is well defined.

$$\forall Y \in \mathfrak{k}_1, p \in \mathfrak{p}_0, [\rho(Y)F, p] = [[Y, F], p] = [Y, [F, p]] - [F, [Y, p]] = [\rho(Y)p, F] \in [\mathfrak{p}_1, \mathfrak{p}_{-2}]$$

and the bracket $[\mathfrak{p}_{-1}, \mathfrak{p}_0] \subset \mathfrak{k}_{-1}$ is well defined.

For $X \in \mathfrak{k}_{-1}, Y \in \mathfrak{k}_1$,

$$\begin{aligned}
[\rho(X)E, \rho(Y)F] &= [X, [E, \rho(Y)F]] - [E, \rho(X)\rho(Y)F] \\
&= [X, [E, \rho(Y)F]] - [E, \rho([X, Y])F] \\
&= -[X, Y] + \alpha([X, Y])H \in \mathfrak{k}_0
\end{aligned}$$

and the bracket $[\mathfrak{p}_1, \mathfrak{p}_{-1}] \subset \mathfrak{k}_0$ is well defined.

$$\forall X, X' \in \mathfrak{k}_{-1},\ [\rho(X)E, \rho(X')E] = [X, [E, \rho(X')E]] - [E, \rho(X)\rho(X')E] = 0,$$

then $[\mathfrak{p}_1, \mathfrak{p}_1] = 0$.

$$\forall Y, Y' \in \mathfrak{k}_1, [\rho(Y)F, \rho(Y')F] = [Y, [F, \rho(Y')F]] - [F, \rho(Y)\rho(Y')F] = 0,$$

then $[\mathfrak{p}_{-1}, \mathfrak{p}_{-1}] = 0$.

The bracket $[\mathfrak{p}_0, \mathfrak{p}_0]$ is then well determined. In fact, for $p, p' \in \mathfrak{p}_0$, as the restriction of ρ to $\mathfrak{k}_0$ is injective, we define $X_0 = [p_0, p_0']$ as the unique element of $\mathfrak{k}_0$ such that $\rho(X_0)$ satisfies :

$$\rho(X_0)E = 0,\ \rho(X_0)F = 0 \text{ and } \rho(X_0)\phi = -[[\phi, p_0], p_0'] - [p_0, [\phi, p_0']]\ \forall \phi \in \mathfrak{p}_{-1} \cup \mathfrak{p}_1$$

and its restriction to $\mathfrak{p}_0$ is then determined because $\mathfrak{p}_0$ is generated by the brackets $[X, p]$ with $X \in \mathfrak{k}_{-1}, p \in \mathfrak{p}_1$, and in this case we have

$$\begin{aligned}
&\rho(X_0)([X, p]) = -[[X, p], X_0] = -[[X, p], [p_0, p_0']] = -[[X, p], p_0], p_0'] - [p_0, [[X, p], p_0']] \\
&= [[[p_0, X], p], p_0'] + [[X, [p_0, p]], p_0'] + [p_0, [[p_0', X], p]] + [p_0, [X, [p_0', p]]] \\
&\in [[[\mathfrak{p}_0, \mathfrak{k}_{-1}], \mathfrak{p}_1], \mathfrak{p}_0] + [[\mathfrak{k}_{-1}, [\mathfrak{p}_0, \mathfrak{p}_1]], \mathfrak{p}_0] + [\mathfrak{p}_0, [[\mathfrak{p}_0, \mathfrak{k}_{-1}], \mathfrak{p}_1]] + [\mathfrak{p}_0, [\mathfrak{k}_{-1}, [\mathfrak{p}_0, \mathfrak{p}_1]]] \\
&\subset [[\mathfrak{p}_{-1}, \mathfrak{p}_1], \mathfrak{p}_0] + [[\mathfrak{k}_{-1}, \mathfrak{k}_1], \mathfrak{p}_0] + [\mathfrak{p}_0, [\mathfrak{p}_{-1}, \mathfrak{p}_1]] + [\mathfrak{p}_0, [\mathfrak{k}_{-1}, \mathfrak{k}_1]] \\
&\subset [\mathfrak{k}_0, \mathfrak{p}_0] + [\mathfrak{k}_0, \mathfrak{p}_0] + [\mathfrak{p}_0, \mathfrak{k}_0] + [\mathfrak{p}_0, \mathfrak{k}_0] \subset [\mathfrak{k}_0, \mathfrak{p}_0] \subset \mathfrak{p}_0.
\end{aligned}$$

Let the vector space $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be equipped with the bracket defined by

$$[X + p, X' + p'] = [X, X']_{\mathfrak{k}} + [p, p'] + \rho(X)p' - \rho(X')p \quad (X, X' \in \mathfrak{k}, p, p' \in \mathfrak{p}).$$

It remains to prove that the Jacobi identity holds. In fact, as $\mathfrak{k}$ is a Lie algebra and ρ is a representation of $\mathfrak{k}$ in $\mathfrak{p}$, then it is clear that for $X, Y, Z \in \mathfrak{k}$,

$$[X, [Y, Z]] = [X, [Y, Z]_{\mathfrak{k}}]_{\mathfrak{k}} = [[X, Y]_{\mathfrak{k}}, Z]_{\mathfrak{k}} + [X, [Y, Z]_{\mathfrak{k}}]_{\mathfrak{k}} = [[X, Y], Z] + [X, [Y, Z]]$$

and for $X, Y \in \mathfrak{k}, p \in \mathfrak{p}$,

$$[X, [Y, p]] = d\kappa(X)d\kappa(Y)p = d\kappa([X, Y]_{\mathfrak{k}})p + d\kappa(Y)d\kappa(X)p = [[X, Y]_{\mathfrak{k}}, p] + [Y, [X, p]].$$

By an other side, the Lie bracket has been defined such that for $X \in \mathfrak{k}, p, p' \in \mathfrak{p}$,

$$[X, [p, p']] = [[X, p], p'] + [p, [X, p']].$$

It remains just to establish the identity

$$[p", [p, p']] = [[p", p], p'] + [p, [p", p']] \quad \forall p, p', p" \in \mathfrak{p} \qquad (*)$$

We prove this identity step by step. Notice first that we can suppose $p" = E$.

In the cases $(p, p' \in \mathfrak{p}_2), (p \in \mathfrak{p}_2, p' \in \mathfrak{p}_1), (p \in \mathfrak{p}_2, p' \in \mathfrak{p}_0), (p \in \mathfrak{p}_2, p' \in \mathfrak{p}_{-1}), (p \in \mathfrak{p}_2, p' \in \mathfrak{p}_{-2}), (p, p' \in \mathfrak{p}_1), (p \in \mathfrak{p}_1, p' \in \mathfrak{p}_0), (p, p' \in \mathfrak{p}_0)$, the identity (*) is trivial.

If $p \in \mathfrak{p}_1, p' \in \mathfrak{p}_{-1}$ then $p = \rho(X)E$ et $p' = \rho(X')F$ with $X \in \mathfrak{k}_{-1}$ and $X' \in \mathfrak{k}_1$, then

$$\begin{aligned}[E, [p, p']] &= [E, [X', X]] - \alpha([X', X])[E, H] \\ &= \rho([X', X])E - 2\alpha([X', X])E \\ &= -\alpha([X', X])E\end{aligned}$$

and as $[[E, p], p'] = 0$ and $[p, [E, p']] = [p, -X'] = \rho(X')d\kappa(X)E = \rho([X', X])E = -\alpha([X', X])E$, the identity (*) is satisfied.

If $p \in \mathfrak{p}_1, p' \in \mathfrak{p}_{-2}$ then $p = \rho(X)E$ with $X \in \mathfrak{k}_{-1}$ and $p' = F$, then $[E, [p, p']] = [E, X] = -\rho(X)E, [[E, p], p'] = 0, [p, [E, p']] = [p, H] = -[H, p] = -\rho(X)E$, then (*) is satisfied.

If $p \in \mathfrak{p}_0, p' \in \mathfrak{p}_{-1}$ then $p' = \rho(X')F$ with $X' \in \mathfrak{k}_1$, then

$$\begin{aligned}[E, [p, p']] &= [E, [p, \rho(X')F]] = [E, [X', [p, F]] - [[X', p], F]] \\ &= [E, [-\rho(X')p, F]] = -[\rho(X')p, [E, F]] = \rho(X')p\end{aligned}$$

and $[[E, p], p'] = 0$ and $[p, [E, p']] = [p, [E, \rho(X')F]] = [X', p] = \rho(X')p$, then (*) is satisfied.

If $p \in \mathfrak{p}_0, p' \in \mathfrak{p}_{-2}$ then $p' = F$ and $[E, [p, p']] = 0, [[E, p], p'] = 0$ and $[p, [E, p']] = -[p, H] = 0$, then (*) is satisfied.

If $p, p' \in \mathfrak{p}_{-1}$ then $p = \rho(X)F$, $p' = \rho(X')F$ with $X, X' \in \mathfrak{k}_1$ and $[E, [p, p']] = 0$, then

$$[[E, p], p'] = [[E, \rho(X)F], \rho(X')F] = \rho(X)\rho(X')F$$

and

$$[p,[E,p']] = [\rho(X)F,[E,\rho(X')F]] = -\rho(X')\rho(X)F$$

and, as $[X,X'] = 0$, then (*) is satisfied.

If $p \in \mathfrak{p}_{-1}, p' \in \mathfrak{p}_{-2}$ then $p = \rho(Y)F$ with $Y \in \mathfrak{k}_1$ and $p' = F$, then $[E,[p,p']] = 0$,

$$[[E,p],p'] = [[E,\rho(Y)F],F] = -[Y,F] = -\rho(Y)F$$

and

$$[p,[E,p']] = [\rho(Y)F,[E,F]] = -[\rho(Y)F,H] = -\rho(Y)F$$

and (*) is satisfied.

If $p, p' \in \mathfrak{p}_{-2}$ then $p = p' = F$, then

$[E,[p,p']] = 0, [[E,p],p'] = [H,F] = -2F$ and $[p,[E,p']] = [p,H] = -[F,H] = -2F$

and (*) is satisfied. $\square$

.

Proposition. (E,F,H) is an sl_2-triple in $\mathfrak{g}$:

$$[H,E] = 2E \quad , \quad [H,F] = -2F \quad , \quad [E,F] = H.$$

$ad(H)$ has eigenvalues $-2,-1,0,1,2$ with respective eigenspaces $\mathfrak{g}_{-2},\mathfrak{g}_{-1},\mathfrak{g}_0,\mathfrak{g}_1,\mathfrak{g}_2$ where

$$\mathfrak{g}_{-2} = \mathfrak{p}_{-2}, \mathfrak{g}_{-1} = \mathfrak{k}_{-1} + \mathfrak{p}_{-1}, \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0, \mathfrak{g}_1 = \mathfrak{k}_1 + \mathfrak{p}_1, \mathfrak{g}_2 = \mathfrak{p}_2.$$

The Lie algebra $\mathfrak{g}$ is 5-graded :

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \quad [\mathfrak{g}_i,\mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

such that

$$\mathfrak{g}_0 = [\mathfrak{g}_{-2},\mathfrak{g}_2] + [\mathfrak{g}_{-1},\mathfrak{g}_1].$$

Moreover, the map $\tau : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\tau(X+p) = \sigma X \sigma + \kappa(\sigma)p \quad (X \in \mathfrak{k}, p \in \mathfrak{p})$$

is an involution of the Lie algebra $\mathfrak{g}$ such that $\tau(\mathfrak{g}_i) = \mathfrak{g}_{-i}$.

$\mathfrak{g}$ will be called the Lie algebra associated to the pair (V,Q).

Theorem 2. $\mathfrak{g}$ is a simple complex Lie algebra.

Proof. Let $\mathcal{I} \neq \{0\}$ be an ideal of $\mathfrak{g}$. For $T \in \mathfrak{g}$, we write $T = T_{\mathfrak{k}} + T_{\mathfrak{p}}$ with $T_{\mathfrak{k}} \in \mathfrak{k}, T_{\mathfrak{p}} \in \mathfrak{p}$. We consider

$$\mathcal{I}_{\mathfrak{p}} = \{p \in \mathfrak{p} \mid \exists T \in \mathcal{I}, T_{\mathfrak{p}} = p\}.$$

$\mathcal{I}_{\mathfrak{p}}$ is a a non trivial $\rho(\mathcal{U}(\mathfrak{k}))$-submodule of $\mathfrak{p}$, then equal to $\mathfrak{p}$. In fact, if $\mathcal{I}_{\mathfrak{p}} = \{0\}$ then $\mathcal{I} \subset \mathfrak{k}$ and then, for each $X \in \mathcal{I}$ and $p \in \mathfrak{p}$, $[X,p] = 0$, which means $\rho(X) = 0$ and then , as ρ is injective (lemma 3), $X = 0$. We deduce that $\mathcal{I}_{\mathfrak{p}} \neq \{0\}$. Now, let be $X \in \mathfrak{k}$ and $p \in \mathcal{I}_{\mathfrak{p}}$, then there exists $T \in \mathcal{I}$ tel que $T_{\mathfrak{p}} = p$. Let be $T = T_{\mathfrak{k}} + p \in \mathfrak{k} + \mathfrak{p}$, then

$$[X,T] = [X,T_{\mathfrak{k}}] + [X,p] = [X,T_{\mathfrak{k}}] + \rho(X)p$$

and as $[X,T] \in \mathcal{I}$, then $[X,p] = \rho(X)p \in \mathcal{I}_{\mathfrak{p}}$. Finally, as , $\mathfrak{p}$ is a simple $\rho(\mathcal{U}(\mathfrak{k}))$-module then $\mathcal{I}_{\mathfrak{p}} = \mathfrak{p}$.

It follows that there exist $T \in \mathcal{I}$ and $T' \in \mathcal{I}$ such that $T_{\mathfrak{p}} = E$ and $T'_{\mathfrak{p}} = F$. We denote by $T = T_{\mathfrak{k}} + E$ and $T' = T'_{\mathfrak{k}} + F$ and with respect to the decomposition $\mathfrak{k} = \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1$ we write

$$T_{\mathfrak{k}} = T_{-1} + T_0 + T_1 \text{ and } T'_{\mathfrak{k}} = T'_{-1} + T'_0 + T'_1.$$

Then

$$[H, T] = -T_{-1} + T_1 + 2E \in \mathcal{I} \text{ and } [H, T'] = -T'_{-1} + T'_1 - 2F \in \mathcal{I}.$$

It follows that

$$S = T + [H, T] = 2T_1 + T_0 + 3E \in \mathcal{I} \text{ and } S' = T' + [H, T'] = 2T'_1 + T'_0 - F \in \mathcal{I}$$
$$[H, S] = 2T_1 + 6E \in \mathcal{I} \text{ and } [H, S'] = 2T'_1 + 2F \in \mathcal{I}$$
$$[H, S] - S = -T_0 + 3E \in \mathcal{I} \text{ and } [H, S'] - S' = -T_0 + 3F \in \mathcal{I}$$

and finally

$$[H, [H, S] - S] = 6E \in \mathcal{I} \text{ and } [H, [H, S'] - S'] = -6F \in \mathcal{I}$$

then

$$E \in \mathcal{I}, F \in \mathcal{I}, H \in \mathcal{I}.$$

We deduce that for $i \neq 0$, $\mathfrak{g}_i \subset \mathcal{I}$. Moreover, as $\mathfrak{p} = \rho(\mathcal{U}(\mathfrak{k}))E$, then $\mathfrak{p} \subset \mathcal{I}$ and in particular $\mathfrak{p}_0 \subset \mathcal{I}$. At last, $\mathfrak{k}_0 = [\mathfrak{k}_{-1}, \mathfrak{k}_1] \subset \mathcal{I}$, and $\mathcal{I} = \mathfrak{g}$. $\square$

Theorem 3.

$$dim\mathfrak{g} = 4dim(V) + 2 + dim\mathfrak{k}_0 + dim\mathfrak{p}_0.$$

Moreover, if $\mathfrak{h}_{\mathfrak{k}}$ is a Cartan subalgebra of $\mathfrak{k}_0$ which contain H, and if

$$\mathfrak{a}_{\mathfrak{p}} = \{p \in \mathfrak{p} \mid [X, p] = 0 \quad \forall X \in \mathfrak{h}_{\mathfrak{k}}\}$$

then $\mathfrak{h} := \mathfrak{h}_{\mathfrak{k}} + \mathfrak{a}_{\mathfrak{p}}$ is a Cartan subalgebra of $\mathfrak{g}$ and

$$rank(\mathfrak{g}) = rank(\mathfrak{k}_0) + dim(\mathfrak{a}_{\mathfrak{p}}).$$

Proof. $\mathfrak{k}_1, \mathfrak{k}_{-1}$ are isomorphic to V. Moreover, $\mathfrak{p}_1$ is generated by the first order partial derivatives of Q, $\mathfrak{p}_0$ is generated by the second order partial derivatives of Q and $\mathfrak{p}_{-1}$, which is genrated by the order 3 partial derivatives of Q, is equal to the space of linear forms on V. Then $\mathfrak{p}_1$ and $\mathfrak{p}_{-1}$ are isomorphic to V. As $\mathfrak{p}_2 = \mathbb{C}.Q$ et $\mathfrak{p}_{-2} = \mathbb{C}.1$, the dimension of $\mathfrak{g}$ is then given by

$$dim\mathfrak{g} = 4dim(V) + 2 + dim\mathfrak{k}_0 + dim\mathfrak{p}_0.$$

Moreover, if $\mathfrak{h}_{\mathfrak{k}}$ is a Cartan subalgebra of $\mathfrak{k}$ containing H, it is a Cartan subalgebra of $\mathfrak{k}_0$ and the subspace of $\mathfrak{p}$, defined by

$$\mathfrak{a}_{\mathfrak{p}} = \{p \in \mathfrak{p} \mid [X, p] = 0 \quad \forall X \in \mathfrak{h}_{\mathfrak{k}}\}$$

is contained in $\mathfrak{p}_0$, and is Abelian. It follows that the subalgebra of $\mathfrak{g}$ given by $\mathfrak{h} := \mathfrak{h}_{\mathfrak{k}} + \mathfrak{a}_{\mathfrak{p}}$ is a Cartan subalgebra of $\mathfrak{g}$ and $rank(\mathfrak{g}) = rank(\mathfrak{k}_0) + dim(\mathfrak{a}_{\mathfrak{p}})$. $\square$

Cartan-Adapted Real Forms

Let $\tilde{\mathfrak{k}}_{\mathbb{R}}$ be the conformal Lie algebra of Euclidean real form J of the Jordan algebra V. Then $\tilde{\mathfrak{k}}_{\mathbb{R}}$ is a real form of $\mathfrak{k}$. Notice that $H \in \tilde{\mathfrak{k}}_{\mathbb{R}}$.

The Cartan involution of $\tilde{\mathfrak{k}}_{\mathbb{R}}$ is given by $\theta : \tilde{\mathfrak{k}}_{\mathbb{R}} \to \tilde{\mathfrak{k}}_{\mathbb{R}}, X \mapsto -\sigma X \sigma$, and the Cartan decomposition of $\tilde{\mathfrak{k}}_{\mathbb{R}}$ writes $\tilde{\mathfrak{k}}_{\mathbb{R}} = \tilde{\mathfrak{k}}^+_{\mathbb{R}} + \tilde{\mathfrak{k}}^-_{\mathbb{R}}$, with

$$\tilde{\mathfrak{k}}^+_{\mathbb{R}} = \{X \in \tilde{\mathfrak{k}}_{\mathbb{R}} \mid \sigma X \sigma = -X\} \text{ and } \tilde{\mathfrak{k}}^-_{\mathbb{R}} = \{X \in \tilde{\mathfrak{k}}_{\mathbb{R}} \mid \sigma X \sigma = X\}.$$

The compact real form $\mathfrak{k}_{\mathbb{R}}$ of $\mathfrak{k}$ is then given by $\mathfrak{k}_{\mathbb{R}} = \tilde{\mathfrak{k}}^+_{\mathbb{R}} + i\tilde{\mathfrak{k}}^-_{\mathbb{R}}$. Notice that $\sigma H \sigma(z) = \frac{d}{dt}_{t=0}\sigma(exp(-t)\sigma(z)) = \frac{d}{dt}_{t=0}exp(t)z = -Hz$, then $\sigma H \sigma = -H$ and $H \in \mathfrak{k}_{\mathbb{R}}$.

Proposition. Let $c_{\mathfrak{k}}$ be the conjugation of $\mathfrak{k}$ with respect to $\mathfrak{k}_{\mathbb{R}}$. It is given by $c_{\mathfrak{k}}(X) = -\sigma \bar{X} \sigma$, where $X \mapsto \bar{X}$ is the conjugation of $\mathfrak{k}$ with respect to $\tilde{\mathfrak{k}}_{\mathbb{R}}$. Moreover $c_{\mathfrak{k}}(\mathfrak{k}_j) = \mathfrak{k}_j$.

Proof. Let $X \in \mathfrak{k}_{\mathbb{R}}$, then $X = X_1 + iX_2$ with $X_1 \in \tilde{\mathfrak{k}}^+_{\mathbb{R}}$ and $X_2 \in \tilde{\mathfrak{k}}^-_{\mathbb{R}}$. Then $-\sigma \bar{X} \sigma = -\sigma \bar{X}_1 \sigma + i\sigma \bar{X}_2 \sigma = X_1 + iX_2 = X$. For $X \in \mathfrak{k}_j$, $[H, c_{\mathfrak{k}}(X)] = c_{\mathfrak{k}}([H, X]) = jc_{\mathfrak{k}}(X)$. $\square$

We denote by $\mathfrak{u}$ the compact real form of $\mathfrak{g}$. It follows that

$$\mathfrak{g} = \mathfrak{u} + i\mathfrak{u} \text{ and } \mathfrak{k}_{\mathbb{R}} = \mathfrak{k} \cap \mathfrak{u}.$$

We put $\mathfrak{p}_{\mathbb{R}} = \mathfrak{p} \cap i\mathfrak{u}$ then the real Lie subalgebra of $\mathfrak{g}$ defined by $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ is a real form of $\mathfrak{g}$. Its Cartan decomposition is just $\mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$. The Cartan signature of $\mathfrak{g}_{\mathbb{R}}$ is then given by

$$s_c = dim(\mathfrak{p}) - dim(\mathfrak{k}) = dim(\mathfrak{p}_0) - dim(\mathfrak{k}_0) + 2.$$

The real form $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ will be called the Cartan-Adapted real form of the Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ associated to the pair (V, Q). (cf. table 1 for the classification)

Corollary. The symmetric pair ($\mathfrak{g}$, $\mathfrak{k}$) is non Hermitian.

Proof. It is a consequence of the fact that the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the complexification of the Cartan decomposition of $\mathfrak{g}$ and the fact that $\mathfrak{p}$ is a simple $\mathfrak{k}$-module. $\square$

Remark. It is possible to see the statement of theorem 1 as a special case of constructions of Lie algebras by Allison and Faulkner, using the Cayley-Dickson process to associate a 5-graded simple Lie algebra to some structurable algebra W, that is $W = \mathfrak{k}_0 + \mathfrak{p}_0$ in our terminology (cf. [A-F]).

(Table 1)

The classification of the simple complex Lie algebras associated to the pair (V, Q) where V is a semisimple Jordan algebra of rank ≤ 4 and Q is a degree 4 semi-inavariant can be obtained by considering all the possible cases for $V = \sum_{i=1}^{s} V_i$ and $Q = \prod_{i=1}^{s} \Delta_i^{k_i}$ where $k_i \in \mathbb{N}^*$ and Δ_i is the polynomial determinant of the simple Jordan algebra V_i. For each case, we determine the dimension and the rank of $\mathfrak{g}$, and the Cartan signature of the real form $\mathfrak{g}_{\mathbb{R}}$.

We obtain by this construction process all the simple real Lie algebras which intersect the minimal nilpotent complex adjoint orbit (cf.[B.]).

V	Q	$\mathfrak{k}$	$\mathfrak{g}$	$\mathfrak{k}_{\mathbb{R}}$	$\mathfrak{g}_{\mathbb{R}}$
$\mathbb{C}$	z^4	$sl(2,\mathbb{C})$	$sl(3,\mathbb{C})$	$su(2)$	$sl(3,\mathbb{R})$
$\mathbb{C}^{p-2}$	$\Delta(z)^2$	$so(p,\mathbb{C})$	$sl(p,\mathbb{C})$	$so(p,\mathbb{R})$	$sl(p,\mathbb{R})$
$\mathbb{C}^{\oplus 2}$	z^2w^2	$sl(2,\mathbb{C})^{\oplus 2}$	$so(6,\mathbb{C})$	$su(2)^{\oplus 2}$	$so(3,3)$
$\mathbb{C}^{\oplus 3}$	z^2uv	$sl(2,\mathbb{C})^{\oplus 3}$	$so(7,\mathbb{C})$	$so(3)^{\oplus 3}$	$so(3,4)$
$\mathbb{C}^{\oplus 4}$	$zuvw$	$sl(2,\mathbb{C})^{\oplus 4}$	$so(8,\mathbb{C})$	$so(3)^{\oplus 4}$	$so(4,4)$
$\mathbb{C}^{p-2} + \mathbb{C}$	$\Delta(z)w$	$so(p,\mathbb{C}) + sl(2,\mathbb{C})$	$so(p+3,\mathbb{C})$	$so(p) + so(3)$	$so(p,3)$
$\mathbb{C}^{p-2} + \mathbb{C}^{\oplus 2}$	$\Delta(z)uv$	$so(p,\mathbb{C}) + sl(2,\mathbb{C})^{\oplus 2}$	$so(p+4,\mathbb{C})$	$so(p) + so(3)^{\oplus 2}$	$so(p,4)$
$\mathbb{C}^{p-2} + \mathbb{C}^{q-2}$	$\Delta(z)\Delta(w)$	$so(p,\mathbb{C}) + so(q,\mathbb{C})$	$so(p+q,\mathbb{C})$	$so(p) + so(q)$	$so(p,q)$
$Sym(4,\mathbb{C})$	$det(z)$	$sp(8,\mathbb{C})$	E_6	$sp(8)$	$E_{6(6)}$
$M(4,\mathbb{C})$	$det(z)$	$sl(8,\mathbb{C})$	E_7	$su(8)$	$E_{7(7)}$
$Asym(8,\mathbb{C})$	$det(z)$	$so(16,\mathbb{C})$	E_8	$so(16)$	$E_{8(8)}$
$Sym(3,\mathbb{C}) + \mathbb{C}$	$det(z)w$	$sp(6,\mathbb{C}) + sl(2,\mathbb{C})$	$\mathfrak{f}_4$	$sp(6) + su(2)$	$F_{4(4)}$
$M(3,\mathbb{C}) + \mathbb{C}$	$det(z)w$	$sl(6,\mathbb{C}) + sl(2,\mathbb{C})$	E_6	$su(6) + su(2)$	$E_{6(2)}$
$Asym(6,\mathbb{C}) + \mathbb{C}$	$det(z)w$	$so(12,\mathbb{C}) + sl(2,\mathbb{C})$	E_7	$so(12) + su(2)$	$E_{7(-5)}$
$Herm(3,\mathbb{O})^{\mathbb{C}} + \mathbb{C}$	$det(z)w$	$E_7 + sl(2,\mathbb{C})$	E_8	$E_7^c + su(2)$	$E_{8(-24)}$
$\mathbb{C}^{\oplus 2}$	z^3w	$sl(2,\mathbb{C})^{\oplus 2}$	G_2	$so(3)^{\oplus 2}$	$G_{2(2)}$

where $p, q \geq 5$.

About the minimal nilpotent orbit

Let G be a connected complex Lie group with Lie algebra $\mathfrak{g}$. Then, the adjoint group G_{ad} of $\mathfrak{g}$ is given by $G_{ad} = G/Z(G)$. Denote by $G_{ad\mathbb{R}}$ the connected subgroup of G_{ad} with Lie algebra $\mathfrak{g}_{\mathbb{R}}$, K_{ad} the connected subgroup of G_{ad} with Lie algebra $\mathfrak{k}$, and by $K_{ad\mathbb{R}}$ the maximal compact subgroup of both $\mathfrak{g}_{\mathbb{R}}$ and K_{ad} with Lie algebra $\mathfrak{k}_{\mathbb{R}}$. It is well known, by the Kostant Sekiguchi correspondance (cf. [S.]), that there is a bijection between the set of nilpotent K_{ad}-orbits in $\mathfrak{p}$ and the set of nilpotent $G_{ad\mathbb{R}}$-orbits in $\mathfrak{g}_{\mathbb{R}}$. Moreover, if $\mathcal{O}_{min}$ is the minimal nilpotent adjoint orbit in $\mathfrak{g}$, then, as $\mathfrak{g}$ is non Hermitian, $\mathcal{O}_{min} \cap \mathfrak{p}$ is equal to the K_{ad}-orbit of the highest weight vector in $\mathfrak{p}$ and $\mathcal{O}_{min} \cap \mathfrak{g}_{\mathbb{R}}$ is the corresponding orbit in $\mathfrak{g}_{\mathbb{R}}$ by the Sekiguchi bijection.
As E is the highest weight vector of the adjoint action of $\mathfrak{k}$ in $\mathfrak{p}$. Then

$$\mathcal{O}_{\mathfrak{p}} := \mathcal{O}_{min} \cap \mathfrak{p} = K_{ad}E.$$

Now, following the terminology of Sekiguchi (cf. [S.]), the sl_2-triple (H, E, F) is normal for the symmetric pair $(\mathfrak{g}, \mathfrak{k})$, that means $H \in \mathfrak{k}, E, F \in \mathfrak{p}$. But the conditions of strict normality, i.e. $H \in i\mathfrak{k}_{\mathbb{R}}, E+F \in \mathfrak{p}_{\mathbb{R}}, i(E-F) \in \mathfrak{p}_{\mathbb{R}}$ (which are equivalent to $\tilde{\theta}(H) = -H$ and $\tilde{\theta}(E) = -F$ where $\tilde{\theta}$ is the Cartan involution of $\mathfrak{g}$) are not satisfied (because $H \in \mathfrak{k}_{\mathbb{R}} \subset \mathfrak{u}$). However, by a lemma of Sekiguchi (cf.[S.]), there exists $k_0 \in K_{ad}$ such that the the normal sl_2-triple $(k_0.H, k_0.E, k_0.F)$ is also strictly normal. Then, the real nilpotent orbit in $\mathfrak{g}_{\mathbb{R}}$ associated to the orbit $K_{ad}.E$ in $\mathfrak{p}$ by the Sekiguchi bijection is given by

$$\mathcal{O}_{\mathbb{R}} := \mathcal{O}_{min} \cap \mathfrak{g}_{\mathbb{R}} = G_{ad\mathbb{R}}.(k_0[\tfrac{1}{2}(E + F + iH)]).$$

A theorem of M. Vergne (cf. [V.]) gives a canonical $K_{ad\mathbb{R}}$-equivariant diffeomorphism (the Vergne-Kronheimer diffeomorphism) from $\mathcal{O}_{\mathbb{R}}$ onto $\mathcal{O}_{\mathfrak{p}}$. (a kind of generalization of the realisation of $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ as $\mathbb{C}^n$.)

In our context, where K is the conformal group and $\tilde{K}^{(2)}$ the meta-conformal group, it is natural to consider the K-orbit $\Xi = \kappa(K).E$ or the $\tilde{K}^{(2)}$-orbit $\tilde{\Xi}^{(2)} = \tilde{\kappa}(\tilde{K}^{(2)}).E$. As the groups K (resp. $\tilde{K}^{(2)}$) and K_{ad} (with same Lie algebra $\mathfrak{k}$) are very closer (they may be equal or one may be a 2-fold covering of the other), and as the stabilizer of E in K (resp. $\tilde{K}^{(2)}$) is equal to $L' \rtimes \sigma N \sigma$, where L' is the kernel of the character χ (resp. $\tilde{\chi}$), then these orbits are finite order coverings of the minimal orbit $\mathcal{O}_{min} \cap \mathfrak{p}$.

Application to Representation Theory

Theorem 4. Let $\tilde{G}_{\mathbb{R}}$ be the connected and simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$. Assume that $(\pi, \mathcal{H})$ is a unitary representation of $K_{\mathbb{R}}$ such that its differential $d\pi$ extends to an irreducible representation $\tilde{\rho}$ of $\mathfrak{g}$ in the space $\mathcal{H}^{K_{\mathbb{R}}}$ of $K_{\mathbb{R}}$-finite vectors. Assume that the operators $\tilde{\rho}(p)$ are antisymmetric for $p \in \mathfrak{p}$. Then there exists a unique unitary irreducible representation $\tilde{\pi}$ of $\tilde{G}_{\mathbb{R}}$ such that $d\tilde{\pi} = \tilde{\rho}$.

Proof. In fact, by the Nelson criterion, it is enough to prove that $\tilde{\rho}(\mathcal{L})$ is essentially self-adjoint for the Laplacian $\mathcal{L}$ of $\mathfrak{g}_{\mathbb{R}}$. Let's consider a basis $\{X_1, \dots, X_k\}$ of $\mathfrak{k}_{\mathbb{R}}$ and a basis $\{p_1, \dots, p_l\}$ of $\mathfrak{p}_{\mathbb{R}}$. As $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ is the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$, then the Laplacian and the Casimir operators of $\mathfrak{g}_{\mathbb{R}}$ are respectively given by

$$\mathcal{L} = X_1^2 + \ldots + X_k^2 + p_1^2 + \ldots + p_l^2$$

and

$$\mathcal{C} = X_1^2 + \ldots + X_k^2 - p_1^2 - \ldots - p_l^2.$$

It follows that $\mathcal{L} = 2(X_1^2 + \ldots + X_k^2) - \mathcal{C}$ and $\tilde{\rho}(\mathcal{L}) = 2\tilde{\rho}(X_1^2 + \ldots + X_k^2) - \tilde{\rho}(\mathcal{C})$.

As $\tilde{\rho}(X_1^2 + \ldots + X_k^2) = d\pi(X_1^2 + \ldots + X_k^2)$ and as π is a unitary representation of $K_{\mathbb{R}}$, then the image $\tilde{\rho}(X_1^2 + \ldots + X_k^2)$ of the Laplacian of $\mathfrak{k}_{\mathbb{R}}$ is essentially self-adjoint. Moreover, $\tilde{\rho}(\mathcal{C})$ is scalar, because the dimension of $\mathcal{H}^{K_{\mathbb{R}}}$ being countable, then the commutant of $\tilde{\rho}$, which is a division algebra over $\mathbb{C}$ has a countable dimension too, and then is equal to $\mathbb{C}$.

It follows that $\tilde{\rho}(\mathcal{L})$ is essentially self-adjoint and that $\tilde{\rho}$ integrates to a unitary representation of $\tilde{G}_{\mathbb{R}}$. $\square$

The case of degree 2 semi-invariant

Suppose that $E = Q$ has degree 2. Then $\chi(h_t) = e^{-t}$ and H defines a graduation on $\mathfrak{p}$ given by

$$\mathfrak{p} = \mathfrak{p}_{-1} + \mathfrak{p}_0 + \mathfrak{p}_1$$

with $\mathfrak{p}_{-1} = \mathbb{C}.F, \mathfrak{p}_1 = \mathbb{C}.E$ and $\mathfrak{p}_0 \simeq V$ is generated by the first order partial derivatives of Q. We consider the vector space $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and, as in the preceding sections, we can prove the following :

Theorem 5. There exists on $\mathfrak{g}$ a unique Lie algebra structure such that

(S_1) $[X, X'] = [X, X']_{\tilde{\mathfrak{k}}}$ $\forall X, X' \in \mathfrak{k}$

(S_2) $[X, p] = \rho(X)p$ $\forall X \in \mathfrak{k}, p \in \mathfrak{p}$

(S_3) $[E, F] = H$.

$\mathfrak{g}$, endowed with this structure, is a simple 3-graded Lie algebra.

Moreover

$$dim\mathfrak{g} = 3dim(V) + 2 + dim(\mathfrak{k}_0)$$

and, if $\mathfrak{h}_\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}_0$ which contains H, and if

$$\mathfrak{a}_\mathfrak{p} = \{p \in \mathfrak{p} \mid [X, p] = 0 \quad \forall X \in \mathfrak{h}_\mathfrak{k}\}$$

then $\mathfrak{h} := \mathfrak{h}_\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$ is a Cartan subalgebra of $\mathfrak{g}$ and

$$rank(\mathfrak{g}) = rank(\mathfrak{k}_0) + dim(\mathfrak{a}_\mathfrak{p}).$$

Moreover, if $\mathfrak{k}_\mathbb{R}$ is the compact real form of $\mathfrak{k}$ and $\mathfrak{u}$ is the compact real form of $\mathfrak{g}$, then $\mathfrak{g} = \mathfrak{u} + i\tilde{\mathfrak{u}}$ is the Cartan decomposition of $\tilde{\mathfrak{g}}$ and $\mathfrak{l}_\mathbb{R} = \mathfrak{u} \cap \mathfrak{k}$. Moreover, if we denote by $\mathfrak{p}_\mathbb{R} = \mathfrak{p} \cap i\mathfrak{u}$, then the real Lie subalgebra of $\mathfrak{g}$ defined by $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} + \mathfrak{p}_\mathbb{R}$ is a real form of $\mathfrak{g}$. Its Cartan decomposition is just $\mathfrak{l}_\mathbb{R} + \mathfrak{p}_\mathbb{R}$. The Cartan signature of $\mathfrak{g}_\mathbb{R}$ is then given by

$$s_c = dim(\mathfrak{p}) - \mathrm{dim}(\mathfrak{k}) = 2 - dim(\mathfrak{k}_0) - dim(V).$$

The real form $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} + \mathfrak{p}_\mathbb{R}$ will be called the Cartan-Adapted real form of the Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ associated to the pair (V, Q). The symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is non Hermitian (cf. table 2 for the classification).

(Table 2)

V	Q	$\mathfrak{k}$	$\mathfrak{g}$	$\mathfrak{k}_\mathbb{R}$	$\mathfrak{g}_\mathbb{R}$
$\mathbb{C}$	z^2	$sl(2,\mathbb{C})$	$so(4,\mathbb{C})$	$su(2)$	$so(3,1)$
$\mathbb{C}^{p-2}$	$\Delta(z)$	$so(p,\mathbb{C})$	$so(p+1,\mathbb{C})$	$so(p)$	$so(p,1)$
$\mathbb{C}+\mathbb{C}$	zw	$sl(2,\mathbb{C})+sl(2,\mathbb{C})$	$so(5,\mathbb{C})$	$su(2)+su(2)$	$so(4,1)$

where $p \geq 5$.

Analogy with the Hermitian case

Now, let V be a semi-simple Jordan algebra with rank r, Q semi-invariant of degree $2r$ and we denote L the structure group of V given by

$$L := \{g \in GL(V) \mid \exists \gamma(g) \in \mathbb{C}, Q(gz) = \gamma(g)Q(z)\}.$$

Let $\mathfrak{p}$ be the complex vector space generated by the polynomials $Q(z-a)$ with $a \in V$. As in the part I, if there exists a character χ of L such that

$$Q(l.z) = \chi(l)^2 Q(z)$$

then the conformal group K of V acts on $\mathfrak{p}$ by the representation κ and if the character χ does not exist, then we consider the action in $\mathfrak{p}$ given by the representation $\tilde{\kappa}$ of the meta-conformal group associated to the semi-invariant Q. We denote by $\rho = d\kappa = d\tilde{\kappa}$. In particular, L acts in $\mathfrak{p}$ by the restriction of κ and for $X \in \mathfrak{l} = Lie(L)$,

$$(\rho(X)p)(z) = \frac{d}{dt}_{t=0}\gamma(exp(\tfrac{1}{2}tX))p(exp(-tX)z).$$

In this case, if $h_t \in K_\circ$ is the dilation of V : $h_t.z = e^{-t}z$ and $H \in \mathfrak{k}_\circ$ the corresponding infinitesimal element, then $\chi(h_t) = e^{-rt}$ and H defines a graduation of $\mathfrak{p}$ given by

$$\mathfrak{p} = \mathfrak{p}_{-r} + \mathfrak{p}_{-r+1} + \ldots + \mathfrak{p}_0 + \ldots + \mathfrak{p}_{r-1} + \mathfrak{p}_r$$

with

$$\mathfrak{p}_j = \{p \in \mathfrak{p} \mid \rho(H)p = jp\}$$

$\mathfrak{p}_j$ is the set of homogeneous polynomials of degree $j+r$ in $\mathfrak{p}$ and in particular,

$$\mathfrak{p}_{-r} = \mathbb{C}, \mathfrak{p}_r = \mathbb{C}.Q, \mathfrak{p}_{-r+1} \simeq \mathfrak{p}_{r-1} \simeq V.$$

We denote by $\mathcal{V}^+ = \mathfrak{p}_{-r+1}, \mathcal{V}^- = \mathfrak{p}_{r-1}$. They are two simple $\mathfrak{l}$-modules. Denote by $\mathcal{V} = \mathcal{V}^+ + \mathcal{V}^-$ and then we consider the complex vector space defined by $\tilde{\mathfrak{g}} = \mathfrak{l} + \mathcal{V}$.

Theorem 6. There exists on $\tilde{\mathfrak{g}}$ a Lie algebra structure such that

(S_1) $\quad [X, X'] = [X, X']_{\tilde{\mathfrak{k}}} \quad \forall X, X' \in \mathfrak{l}$

(S_2) $\quad [X, p] = \rho(X)p \quad \forall X \in \mathfrak{l}, p \in \mathcal{V}$.

$\tilde{\mathfrak{g}}$, endowed with this structure, is a simple 3-graded Lie algebra.

Moreover

$$dim\mathfrak{g} = 2dim(V) + dim(\mathfrak{l})$$

and, if $\mathfrak{h}_{\mathfrak{k}}$ is a Cartan subalgebra of $\mathfrak{l}$, then it is a Cartan subalgebra of $\tilde{\mathfrak{g}}$ and $rank(\tilde{\mathfrak{g}}) = rank(\tilde{\mathfrak{k}})$.

Proof. In fact, as for each $p \in \mathfrak{p}_{r-1}$ and for each $p' \in \mathfrak{p}_{-r+1}$, there exists a unique $X \in \mathfrak{k}_{-1}$ and a unique $X' \in \mathfrak{k}_1$ such that $p = \rho(X)F$ and $p' = \rho(X')E$ (where $F = 1$ and $E = Q$), then we define $[p, p'] = [X, X'] \in \mathfrak{l}$. Also, for $p_1 = \rho(X_1)E$ and $p_2 = \rho(X_2)E$ (with $X_1, X_2 \in \mathfrak{k}_{-1}$), we define the bracket as $[p_1, p_2] = [X_1, X_2] = 0$ and for $p'_1 = \rho(X'_1)F$ and $p'_2 = \rho(X'_2)F$ (with $X'_1, X'_2 \in \mathfrak{k}_1$), we define the bracket as $[p'_1, p'_2] = [X'_1, X'_2] = 0$. It becomes then easy to show that the Jacobi identity holds and that $\tilde{\mathfrak{g}}$ is isomorphic to the conformal Lie algebra of V. $\square$

Moreover, denote by $\mathfrak{l}_{\mathbb{R}}$ the compact real form of $\mathfrak{l}$ and by $\tilde{\mathfrak{u}}$ the compact real form of $\tilde{\mathfrak{g}}$. Then $\tilde{\mathfrak{g}} = \tilde{\mathfrak{u}} + i\tilde{\mathfrak{u}}$ is the Cartan decomposition of $\tilde{\mathfrak{g}}$ and $\mathfrak{l}_{\mathbb{R}} = \tilde{\mathfrak{u}} \cap \mathfrak{l}$. Moreover, if we denote by $\mathcal{V}_{\mathbb{R}} = \mathcal{V} \cap i\tilde{\mathfrak{u}}$, then the real Lie subalgebra of $\tilde{\mathfrak{g}}$ defined by $\tilde{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{l}_{\mathbb{R}} + \mathcal{V}_{\mathbb{R}}$ is a real form of $\tilde{\mathfrak{g}}$. Its Cartan decomposition is just $\mathfrak{l}_{\mathbb{R}} + \mathcal{V}_{\mathbb{R}}$. The Cartan signature of $\tilde{\mathfrak{g}}_{\mathbb{R}}$ is then given by

$$s_c = dim(\mathcal{V}) - \dim(\mathfrak{l}) = 2dim(V) - dim(\mathfrak{l}).$$

The real form $\tilde{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{l}_{\mathbb{R}} + \mathcal{V}_{\mathbb{R}}$ will be called the Cartan-Adapted real form of the Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{l} + \mathcal{V}$ associated to the pair (V, Q). (cf. table 3 for the classification)

Corollary. The real forms $\tilde{\mathfrak{g}}_{\mathbb{R}}$ are of Hermitian type.

Proof. It is a consequence of the fact that the decomposition $\tilde{\mathfrak{g}} = \mathfrak{l} + \mathcal{V}$ is the complexification of the Cartan decomposition of $\tilde{\mathfrak{g}}_{\mathbb{R}}$ and the fact that $\mathcal{V}$ is a sum of two irreducible representations, $\mathcal{V}^+$ and $\mathcal{V}^-$ of $\mathfrak{l}$. $\square$

(Table 3)

V	Q	$\mathfrak{l}$	$\tilde{\mathfrak{g}}$	$\mathfrak{l}_{\mathbb{R}}$	$\tilde{\mathfrak{g}}_{\mathbb{R}}$
$\mathbb{C}$	z^2	$\mathbb{C}$	$sl(2,\mathbb{C})$	$i\mathbb{R}$	$sl(2,\mathbb{R})$
$\mathbb{C}^{p-2}$	$\Delta(z)^2$	$so(p-2,\mathbb{C}) + \mathbb{C}$	$so(p,\mathbb{C})$	$so(p-2) + i\mathbb{R}$	$so(p-2,2)$
$Sym(r,\mathbb{C})$	$det(z)^2$	$sl(r,\mathbb{C}) + \mathbb{C}$	$sp(r,\mathbb{C})$	$su(r) + i\mathbb{R}$	$sp(r,\mathbb{R})$
$M(r,\mathbb{C})$	$det(z)^2$	$sl(r,\mathbb{C})^{\oplus 2} + \mathbb{C}^{\oplus 2}$	$sl(2r,\mathbb{C})$	$su(r)^{\oplus 2} + i\mathbb{R}^{\oplus 2}$	$sl(2r,\mathbb{R})$
$Asym(2r,\mathbb{C})$	$det(z)$	$sl(2r,\mathbb{C}) + \mathbb{C}$	$so(4r,\mathbb{C})$	$so(2r)$	$so^*(4r)$
$Herm(3,\mathbb{O})^{\mathbb{C}}$	$det(z)^2$	$E_6(\mathbb{C})^{\mathbb{C}}$	$E_7(\mathbb{C})$	$E_6(\mathbb{R}) + i\mathbb{R}$	$E_{7(-25)}$

where $p \geq 5$.

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